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Color groups arising from index-n subgroups of symmetry groups

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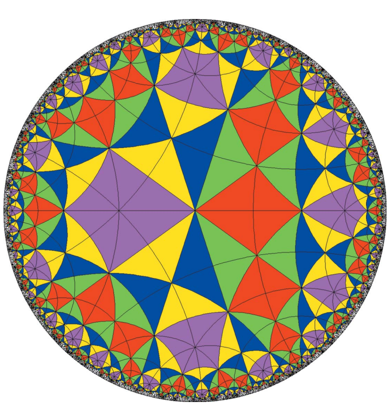
One of the main goals in the study of color symmetry is to classify colorings of symmetrical objects through their color groups. The term color group is taken to mean the subgroup of the symmetry group of the uncolored symmetrical object which induces a permutation of colors in the coloring. This work looks for methods of determining the color group of a colored symmetric object. It begins with an index n subgroup H of the symmetry group G of the uncolored object. It then considers H-invariant colorings of the object, so that the color group H^* will be a subgroup of G containing H. In other words, $H \leq H^* \leq G$. It proceeds to give necessary and sufficient conditions for the equality of H^* and G. If $H^* \neq G$ and n is prime, then $H^* = H$. On the other hand, if $H^* \neq G$ and n is not prime, methods are discussed to determine whether H^* is G, H or some intermediate subgroup between H and G.



1. Introduction

Colorings of a symmetrical object reveal properties of the object which may not be immediately recognizable. For example, Fig. 1 exhibits a coloring of the vertices of a regular icosahedron that reveals how the vertices may be partitioned to form four equilateral triangles. In the figure, vertices of the same color form an equilateral triangle.

Color symmetry can also give a visual representation of some group-theoretical ideas. For instance, in a colored symmetrical object, one can talk of the stabilizer of a certain color with respect to the symmetry group of the object.

Roth (1982) presented the basic ideas about perfect and non-perfect colorings, and gave some simple examples and illustrations. One of the topics he discussed in his paper is the idea of equivalent colorings, which he illustrated in Roth (1985) where he colored a pattern with symmetry group $p4m$.

Schwarzenberger's (1984) work is a compilation of more than 50 years' worth of results in color symmetry, including those in Roth (1982). In the paper, he mentioned some of the fundamental results of color symmetry, and went on to say that even with only a few results, color symmetry has produced a lot of articles, especially in crystallography journals. He also said that the development of color symmetry reflects the development of mathematics in general, and may be useful when studying the history of mathematics.

Senechal (1988) mentioned some of the applications of color symmetry, more specifically in the study of crystal formation, along with some of the open problems. She stated that the basic problem has always been the classification of colorings. One of the classification methods she proposed was the classification using the subgroup of the symmetry group of the uncolored object which induces a permutation of colors in the coloring. This paper makes use of this classification

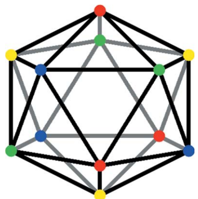


Figure 1
A coloring of the vertices of a regular icosahedron.

method and refers to this subgroup as the *color group* of the coloring. Senechal noted, however, that such a method is not enough to completely classify colored patterns, as two completely different colorings may still have the same color group.

De Las Peñas & Felix (1997) considered colored patterns where the color group is a subgroup of the symmetry group of the uncolored pattern of index at most 3. The paper used a framework (De Las Peñas *et al.*, 1999) where the uncolored pattern is the G-orbit of a subset S of a fundamental domain for the symmetry group G. In this case, the elements of the G-orbit of S may be labeled using the elements of G, and the stabilizer subgroup of S in G is trivial. In a subsequent paper, colorings based on a subgroup of index 4 in the symmetry group were studied (De Las Peñas & Felix, 2003).

Succeeding papers (De Las Peñas *et al.*, 2006, 2011; Gozo, 2010) reduced the restrictions on the patterns being colored. These papers considered patterns whose tiles did not form a single G-orbit. This problem was attacked by coloring each G-orbit separately. More importantly, they also considered patterns where the tiles were not subsets of a fundamental domain of the symmetry group G, so that the stabilizer subgroup of a tile need not be trivial.

For instance, a method for obtaining colorings of tilings of the hyperbolic plane where the associated color group is equal to the full symmetry group was discussed in De Las Peñas *et al.* (2006). Such colorings are called perfect colorings. On the other hand, a method for identifying the color groups arising from index-2 subgroups of symmetry groups was outlined in De Las Peñas *et al.* (2011). This method was then applied to tilings of the hyperbolic plane to obtain various colorings, some of which are perfect, and others with associated color groups of index 2 in the symmetry group (called semi-perfect colorings; Felix & Loquias, 2008).

As an extension, Gozo's dissertation (Gozo, 2010) provided a similar method for determining the color groups arising from index-3 and -4 subgroups of symmetry groups. Here, the colorings obtained have different associated color groups, with indices ranging from 1 to 4.

In this paper, we give a framework for determining the color group associated with colorings arising from index-n subgroups of symmetry groups. The main result of the paper is theorem 2, which states how to determine if a coloring arising from a subgroup H of any index in G is perfect. §3 also discusses how this theorem is useful to identify the color group of a given coloring, and how theorem 2 completely solves the problem of determining the color group when the index of H in G is prime. We use index-5 subgroups as examples for the prime case, while index-6 subgroups are used for the case where n is not prime. §4 provides examples of colorings arising from such subgroups of various symmetry groups, ranging from colorings of regular polygons to colorings of tilings of the hyperbolic plane.

2. Preliminaries

Let G be the symmetry group of a set X. Consider a partition of X given by $P = \{P_1, P_2, \ldots, P_n\}$. The partition P is said to be a *coloring* of X, where each P_i is called a *color*. To put it another way, two elements of X are assigned the same color if and only if they belong to the same set P_i in the partition P.

An element $g \in G$ is said to *permute the colors* or *induce a permutation of colors* in a coloring P of X if g leaves the partition P invariant, *i.e.* $gP = P$. The set of elements of G which induce a permutation of colors form a subgroup of G. This subgroup $\{g \in G \mid gP = P\}$ is called the *color group* associated with the coloring P. If the color group is G, we say that the coloring is *perfect*. Otherwise, the coloring is said to be *non-perfect*.

Fig. 2 shows eight isosceles right triangles which are arranged to form a square. When uncolored, the symmetry group G of the set of eight triangles is isomorphic to D_4, the dihedral group with eight elements. It is easy to check that the color group of the coloring in Fig. 2 is G, so that the coloring is perfect. For example, the 90° counterclockwise rotation about the center of the square sends green to blue, blue to red, red to yellow and yellow back to green. On the other hand, the reflection along the horizontal line passing through the center interchanges green and blue, as well as red and yellow.

Fig. 3 makes use of the same set of eight triangles with different colors. In this case, the color group is not G. Take, for example, the reflection along the diagonal going through the lower left corner and the upper right corner of the square. Applying this reflection, we can see that some green triangles go to green triangles, while other green triangles go to red triangles. This means that the said reflection does not induce a permutation of colors. The color group of the coloring is the

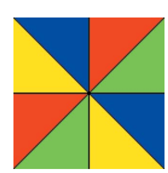


Figure 2
A perfect coloring.

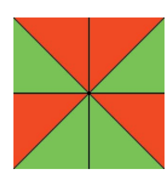


Figure 3
A non-perfect coloring.

subgroup of G generated by the half-turn about the center of the square and the reflection along the horizontal line.

3. Framework for the colorings

Let G be the symmetry group of a set $\mathcal{X}$. The set $\mathcal{X}$ will be partitioned into G-orbits, and each G-orbit will be colored separately.

Let $x_1 \in \mathcal{X}$, and take $X = Gx_1 = \{gx_1 \mid g \in G\}$, the G-orbit of x_1, so that G acts transitively on the set X. We have the following framework for coloring:

(i) A coloring of X for which G permutes the colors satisfies the following property: 'for every $x \in X$, there exists $H \le G$ such that $\mathrm{Stab}_G\, x \le H$ and the coloring is described by the partition $\{gHx \mid g \in G\}$'.

(ii) If $x \in X$ and $H \le G$ such that $\mathrm{Stab}_G\, x \le H$ and $[G : H] = n$, then $P = \{gHx \mid g \in G\}$ is an index-n coloring of X for which G permutes the colors.

Now, let $H < G$, where $[G : H] = n$. In this case, X will be partitioned into m H-orbits with $1 \le m \le n$ so that $X = Gx_1 = \bigcup_{i=1}^{m} Hx_i$, where $x_i \in X$, and $x_k \notin Hx_j$ whenever $j \ne k$. We look at colorings of X arising from the subgroup H and the color groups associated with the said colorings.

We begin with H-invariant colorings of X. Because of the first statement of the framework above, any H-invariant coloring of X may be written in the form $P = \bigcup_{i=1}^{s}\{hJ_iX_i \mid h \in H\}$ such that $\forall x \in X_i$, $\mathrm{Stab}_H\, x \le J_i \le H$, for $1 \le i \le s$, and where J_iX_i is defined as $J_iX_i := \bigcup_{x\in X_i} J_ix$.

We now formulate a general theorem for determining whether the color group is G. Partition $\{x_j\}_{j=1}^{m}$ as $\bigcup_{i=1}^{s} X_i$, where $1 \le s \le m$, and without loss of generality, suppose $x_i \in X_i$ for all i, $1 \le i \le s$. Form the partition P of X given by $P = \bigcup_{i=1}^{s}\{hJ_iX_i \mid h \in H\}$ such that $\forall x \in X_i$, $\mathrm{Stab}_H\, x \le J_i \le H$, for $1 \le i \le s$. A partition of this form will produce a coloring of X with c colors, where c is given by $c = \sum_{i=1}^{s}[H : J_i]$. This is due to the fact that each $\{hJ_iX_i \mid h \in H\}$ will give rise to a number of colors equal to the index of J_i in H.

Note that, for partitions of this form, if $h' \in H$, we have

$$\begin{aligned} h'P &= \bigcup_{i=1}^{s}\{h'hJ_iX_i \mid h \in H\} \\ &= \bigcup_{i=1}^{s}\{h''J_iX_i \mid h'' \in H\} \\ &= P, \end{aligned}$$

so that P is H-invariant, *i.e.* for all $h \in H$, $hP = P$. Therefore, H is always a subgroup of the color group of P. We will now call the color group H^*. We have $H \le H^* \le G$.

We will make use of the following lemma.

Lemma 1. Let G be a group acting on a set X, and let $g \in G$, $x_1, x_2 \in X$. If $gx_1 = x_2$, then $\mathrm{Stab}_G\, x_2 = g(\mathrm{Stab}_G\, x_1)g^{-1}$.

Proof. First, we show that $\mathrm{Stab}_G\, x_2 \subseteq g(\mathrm{Stab}_G\, x_1)g^{-1}$. Let $g' \in \mathrm{Stab}_G\, x_2$, *i.e.* $g'x_2 = x_2$. Since $gx_1 = x_2$, then $g'(gx_1) = gx_1$. Therefore, $g^{-1}g'gx_1 = x_1$ so that $g^{-1}g'g \in \mathrm{Stab}_G\, x_1$. Therefore, $g' \in g(\mathrm{Stab}_G\, x_1)g^{-1}$.

Next, we show the reverse inclusion, $g(\mathrm{Stab}_G\, x_1)g^{-1} \subseteq \mathrm{Stab}_G\, x_2$. Let $g' \in g(\mathrm{Stab}_G\, x_1)g^{-1}$. Therefore, g' is of the form gg^*g^{-1}, where $g^* \in \mathrm{Stab}_G\, x_1$. So we have $g'x_2 = gg^*g^{-1}x_2 = gg^*x_1 = gx_1 = x_2$. Hence, $g' \in \mathrm{Stab}_G\, x_2$ and so $g(\mathrm{Stab}_G\, x_1)g^{-1} \subseteq \mathrm{Stab}_G\, x_2$. □

Next we construct a complete set of right coset representatives for H in G. Fix $y_1 = e$. Since G acts transitively on X, we can find $y_2, \ldots, y_m \in G\backslash H$ so that $y_ix_1 = x_i$ for $i = 1, \ldots, m$. Note that $Hy_j \ne Hy_k$ for $1 \le j, k \le m$. Indeed, if $Hy_j = Hy_k$ (with $j \ne k$), then $Hy_jx_1 = Hy_kx_1$, and so $Hx_j = Hx_k$. Thus, $x_k \in Hx_j$, a contradiction. Extend the set $\{y_1, \ldots, y_m\}$ to a complete set $\{y_1, \ldots, y_m, g_{m+1}, \ldots, g_n\}$ of right coset representatives of H in G.

Let $m + 1 \le j \le n$. We have $g_jx_1 = h_jx_i$ for some $h_j \in H$ and where $1 \le i \le m$. Set $y_j := h_j^{-1}g_j \in Hg_j$, so that $Y = \{y_i\}_{i=1}^{n}$ is a complete set of right coset representatives of H in G and $y_jx_1 = x_i$ for some i, $1 \le i \le m$.

Partition Y as $Y = \bigcup_{i=1}^{s} Y_i$ where $Y_i = \{y_j \in Y \mid y_jx_1 \in X_i\}$. Note that this partition is induced by the partition of $\{x_j\}_{j=1}^{m}$. Let $S_i = \bigcup_{y\in Y_i} J_iyy_i^{-1}$, for $1 \le i \le s$.

Theorem 2. In a coloring of X induced by the partition $P = \bigcup_{i=1}^{s}\{hJ_iX_i \mid h \in H\}$ given above, $H^* = G$ if and only if $S_1 \le G$ and $\forall y \in Y_i$, $yS_1y^{-1} = S_i$, for $2 \le i \le s$.

Proof. Assume that $H^* = G$. First, we show that $S_1 \le G$.

Let $y \in Y_1$, and so $yx_1 \in X_1$. Since $H^* = G$, we have $yP = P$.

Now, $yx_1 \in yJ_1X_1$, but since $yx_1 \in X_1$, we also have $yx_1 \in J_1X_1$. Since $yP = P$, we obtain $yJ_1X_1 = J_1X_1$.

We show that $S_1 = \mathrm{Stab}_G\, J_1X_1$. Let $g \in S_1$. Then g is of the form jyy_1^{-1}, where $j \in J_1$ and $y \in Y_1$. Since $y_1 = e$, then $g = jy$. We have $gJ_1X_1 = jyJ_1X_1 = jJ_1X_1 = J_1X_1$. Therefore, $g \in \mathrm{Stab}_G\, J_1X_1$, and so $S_1 \subseteq \mathrm{Stab}_G\, J_1X_1$.

We now show the reverse inclusion $\mathrm{Stab}_G\, J_1X_1 \subseteq S_1$. Let $g \in \mathrm{Stab}_G\, J_1X_1$, *i.e.* $gJ_1X_1 = J_1X_1$. We have $gx_1 = jx$ for some $j \in J_1$, $x \in X_1$. Now, since $g \in G$, then g is of the form hy for some $h \in H$ and $y \in Y$, so we have $hyx_1 = jx$. We now proceed to show that $h \in J_1$ and $y \in Y_1$. We have the following equation:

$$yx_1 = h^{-1}jx. \tag{1}$$

Now, $h^{-1}j \in H$, so $yx_1 \in Hx$. But from the selection of y's, we have

$$yx_1 = x. \tag{2}$$

But $x \in X_1$ and, therefore, $y \in Y_1$.

From equations (1) and (2), we get $x = h^{-1}jx$, or equivalently, $j^{-1}hx = x$. Therefore, $j^{-1}h \in \mathrm{Stab}_H\, x$. Since $x \in X_1$, from the given we have $\mathrm{Stab}_H\, x \le J_1$. Therefore, $j^{-1}h \in J_1$, so $h \in J_1$.

We have now shown that g is of the form hy, where $h \in J_1$ and $y \in Y_1$. Again, since $y_1 = e$, then $g = hyy_1^{-1}$. Therefore, $g \in S_1$ and so $\mathrm{Stab}_G\, J_1X_1 \subseteq S_1$. Hence, $S_1 = \mathrm{Stab}_G\, J_1X_1$ and $S_1 \le G$.

Next we show that $\forall y \in Y_i$, $yS_1y^{-1} = S_i$, for $2 \le i \le s$. Fix i. Let $y \in Y_i$, so that $yx_1 \in X_i$, and therefore, $yx_1 \in J_iX_i$.

However, $yx_1 \in yJ_1X_1$ as well. Since $H^* = G$, then $yP = P$. We have

$$yJ_1X_1 = J_iX_i. \quad (3)$$

From lemma 1, we get $\mathrm{Stab}_G J_iX_i = y(\mathrm{Stab}_G J_1X_1)y^{-1}$. It has already been proven that $S_1 = \mathrm{Stab}_G J_1X_1$, so we have

$$\mathrm{Stab}_G J_iX_i = yS_1y^{-1}. \quad (4)$$

It remains to show that $S_i = \mathrm{Stab}_G J_iX_i$.

First, we show that $S_i \subseteq \mathrm{Stab}_G J_iX_i$. Let $g \in S_i$. Then g is of the form jyy_i^{-1}, where $j \in J_i$ and $y \in Y_i$, so we have

$$\begin{aligned} gJ_iX_i &= jyy_i^{-1}J_iX_i \\ &= jyJ_1X_1 \text{ [as a result of (3)]} \\ &= jJ_iX_i \\ &= J_iX_i. \end{aligned}$$

Therefore, $g \in \mathrm{Stab}_G J_iX_i$.

Next, we show the reverse inclusion $\mathrm{Stab}_G J_iX_i \subseteq S_i$. Let $g \in \mathrm{Stab}_G J_iX_i$, *i.e.* $gJ_iX_i = J_iX_i$. Therefore, for some $j \in J_i, x \in X_i$:

$$gx_i = jx. \quad (5)$$

Now, $G = \bigcup_{y \in Y} Hy$, and therefore, $Gy_i^{-1} = \bigcup_{y \in Y} Hyy_i^{-1}$. But $Gy_i^{-1} = G$, so the set $Yy_i^{-1} = \{y_1y_i^{-1}, \ldots, y_ny_i^{-1}\}$ is also a complete set of right coset representatives of H in G. Therefore, g is of the form hyy_i^{-1} for some $h \in H$ and $y \in Y$.

We now proceed to show that $y \in Y_i$ and $h \in J_i$. From equation (5), we get $hyy_i^{-1}x_i = jx$, or equivalently,

$$yx_1 = h^{-1}jx. \quad (6)$$

Now, $h^{-1}j \in H$, so $yx_1 \in Hx$. But from the selection of y's, we have

$$yx_1 = x. \quad (7)$$

But $x \in X_i$, so we get $y \in Y_i$.

From equations (6) and (7), we get $x = h^{-1}jx$, or equivalently, $j^{-1}hx = x$. Therefore, $j^{-1}h \in \mathrm{Stab}_H x$. Since $x \in X_i$, then $\mathrm{Stab}_H x \leq J_i$, and so $j^{-1}h \in J_i$. Therefore, $h \in J_i$.

We have now shown that g is of the form hyy_i^{-1}, where $h \in J_i$ and $y \in Y_i$. Therefore, $g \in S_i$, and so $\mathrm{Stab}_G J_iX_i \subseteq S_i$. We conclude that $S_i = \mathrm{Stab}_G J_iX_i$. From this fact and equation (4), we have $S_i = yS_1y^{-1}$.

Conversely, suppose that $S_1 \leq G$ and $\forall y \in Y_i$, $yS_1y^{-1} = S_i$, for $2 \leq i \leq s$. Note that since $S_1 \leq G$, then $\forall y \in Y_1 \subseteq S_1$, $yS_1y^{-1} = S_1$ as well. That is, the second condition also works when $i = 1$. From the definition of S_1, we have

$$S_1x_1 = \bigcup_{y \in Y_1} J_1yx_1 = J_1X_1.$$

Similarly, for $2 \leq i \leq s$ we have

$$S_ix_i = \bigcup_{y \in Y_i} J_iyy_i^{-1}x_i = \bigcup_{y \in Y_i} J_iyx_1 = J_iX_i.$$

So we rewrite P as follows:

$$P = \bigcup_{i=1}^{s} \{hJ_iX_i | h \in H\} = \bigcup_{i=1}^{s} \{hS_ix_i | h \in H\}.$$

Since $S_1 \leq G$, then $\forall y \in Y_i$, $S_i = yS_1y^{-1}$ is also a subgroup of G for $2 \leq i \leq s$. Also, $yy_i^{-1} \in S_i$ for all $y \in Y_i$, so $S_iyy_i^{-1} = S_i$. We then have

$$\begin{aligned} P &= \bigcup_{i=1}^{s} \{hS_ix_i \mid h \in H\} \\ &= \bigcup_{i=1}^{s} \{hS_iyy_i^{-1}x_i \mid h \in H, y \in Y_i\} \\ &= \bigcup_{i=1}^{s} \{hS_iyx_1 \mid h \in H, y \in Y_i\}. \end{aligned}$$

Now, $\forall y \in Y_i$, $yS_1y^{-1} = S_i$, for $1 \leq i \leq s$, or equivalently, $yS_1 = S_iy$. We get

$$\begin{aligned} P &= \bigcup_{i=1}^{s} \{hS_iyx_1 \mid h \in H, y \in Y_i\} \\ &= \bigcup_{i=1}^{s} \{hyS_1x_1 \mid h \in H, y \in Y_i\} \\ &= \{hyS_1x_1 \mid h \in H, y \in Y\}. \end{aligned}$$

But Y is a complete set of right coset representatives for H in G. Therefore

$$P = \{gS_1x_1 \mid g \in G\}.$$

So if $g' \in G$, then $g'P = P$. Therefore, $H^* = G$. □

Theorem 2 is a generalization of theorems formulated in De Las Peñas *et al.* (2011) and Gozo (2010). For instance, if H is of index 2 in G and $Y = \{e, y\}$, then theorem 3 from De Las Peñas *et al.* (2011) is obtained by choosing $S_1 = J_1$ and $S_2 = J_2$ in theorem 2. Note that since $S_1 = J_1$, then S_1 is automatically a subgroup of G, so the first condition need not be written in this case.

As an example, we consider the case where $[G : H] = 6$ and $X = Gx_1 = Hx_1 \cup Hx_2 \cup Hx_3 \cup Hx_4$. Let $y_1 = e$ (so that $y_1x_1 = x_1$). Since $x_2, x_3, x_4 \in X = Gx_1$, then we can find $y_2, y_3, y_4 \in G\backslash H$ such that $y_2x_1 = x_2$, $y_3x_1 = x_3$ and $y_4x_1 = x_4$. Note that Hy_2, Hy_3 and Hy_4 are different right cosets because if $Hy_j = Hy_k$ (with $j \neq k$), then $Hx_j = Hx_k$, and so $x_k \in Hx_j$, a contradiction. Take $g_5, g_6 \in G$ such that y_1, y_2, y_3, y_4, g_5 and g_6 form a complete set of coset representatives of H in G.

Since $Gx_1 = Hx_1 \cup Hx_2 \cup Hx_3 \cup Hx_4$, then for $j = 5, 6$, $g_jx_1 = h_jx_i$ for some $h_j \in H$ and $i, 1 \leq i \leq 4$. Take $y_j := h_j^{-1}g_j \in Hg_j$, so that $Y = \{y_i\}_{i=1}^{6}$ is a complete set of right coset representatives of H in G, and $y_jx_1 = x_i$ for some $i, 1 \leq i \leq 4$. Table 1 summarizes the possible X_i's and the corresponding sufficient and necessary requirements so that $H^* = G$.

We now proceed to the discussion on how to implement theorem 2 to obtain the color group of a coloring satisfying the requirements given previously. The process for determining color groups is as follows. They are similar to those used by Gozo (2010).

(i) Take $H < G$ such that $[G : H] = n$.

Table 1
The possible X_i's and the corresponding sufficient and necessary requirements so that $H^* = G$.

X_i's	$H^* = G$ if and only if
$X_1 = \{x_1\}$,	$S_1 \le G$ and
$X_2 = \{x_2\}$,	$\forall y \in Y_2,\ yS_1y^{-1} = S_2$
$X_3 = \{x_3\}$,	$\forall y \in Y_3,\ yS_1y^{-1} = S_3$
$X_4 = \{x_4\}$	$\forall y \in Y_4,\ yS_1y^{-1} = S_4$
$X_1 = \{x_1, x_4\}$,	$S_1 \le G$ and
$X_2 = \{x_2\}$,	$\forall y \in Y_2,\ yS_1y^{-1} = S_2$
$X_3 = \{x_3\}$	$\forall y \in Y_3,\ yS_1y^{-1} = S_3$
$X_1 = \{x_1, x_4\}$,	$S_1 \le G$ and
$X_2 = \{x_2, x_3\}$	$\forall y \in Y_2,\ yS_1y^{-1} = S_2$
$X_1 = \{x_1, x_3, x_4\}$,	$S_1 \le G$ and
$X_2 = \{x_2\}$	$\forall y \in Y_2,\ yS_1y^{-1} = S_2$
$X_1 = \{x_1, x_2, x_3, x_4\}$	$S_1 \le G$

(ii) Color X using a partition P as was used in theorem 2, so that $H \le H^* \le G$.

(iii) Use theorem 2 to determine if $H^* = G$.

(iv) If $H^* \ne G$ and n is prime, then $H^* = H$.

(v) If $H^* \ne G$ and n is not prime, use the right coset representatives of H in G to determine H^*.

The idea behind the fifth step is as follows. If $H^* \ne G$, we consider the right coset representatives of H in G. Here, we choose one of the right coset representatives to be e, the identity. Since $H^* \ne G$, then not all of the right coset representatives will induce a permutation of colors, and we proceed by checking which of them do.

If a right coset representative y induces a permutation of colors, then the whole coset Hy induces a permutation of colors, since all the elements of H do by default, since P is H-invariant. To obtain H^*, we simply take the union of those right cosets whose representatives permute the colors. If none of the five non-identity right coset representatives permute the colors, then $H^* = H$.

Note that when the index of H in G is a prime number, theorem 2 completely solves the problem of determining the color group, due to the fact that there can be no intermediate subgroups between H and G when the index of H in G is prime. Thus, if the color group is not G, then it is definitely H.

4. Examples

The following examples show different types of colorings, with color groups of varying indices in the symmetry group of the object. For each example, written are the subgroup H of the symmetry group G, the set Y, which is a complete set of right coset representatives of H in G which satisfies the requirements given in the previous discussions, the partition P used to color the object and a brief discussion on the color group H^* and how it was obtained. For the first few examples, the symmetry group G acts transitively on the set X of objects to be colored. We write $X = Gx_1$, where x_1 is as indicated. On the other hand, for the last example, the symmetry group G does not act transitively on the set. There we illustrate some interesting observations on such colorings.

4.1. Regular icosahedron

We begin with Fig. 1, the regular icosahedron shown in §1. A regular icosahedron has symmetry group $G \cong A_5 \times C_2$ of order 120. We will take the set X to be the set of vertices of the icosahedron, which form one orbit under G. To obtain a simple notation for the elements of G, we number the faces of the icosahedron using the numbers 1, 2, 3, 4 and 5 in such a way that no two adjacent faces are assigned the same number. The numbering of the faces is given in Fig. 4.

Upon numbering the faces, we can now express the elements of G using the effect they induce on the numbers on the faces. For example, the 120° counterclockwise rotation whose axis passes through the center of the front face labeled 4 and the center of the icosahedron is denoted (123), since it sends 1 to 2, 2 to 3 and 3 to 1, while it sends 4 to 4 and 5 to 5.

We now go back to the coloring. For the coloring, we have $H = \langle(123), (124)\rangle = \{(1), (12)(34), (13)(24), (14)(23), (123), (132), (124), (142), (134), (143), (234), (243)\}$, which is a group with 12 elements isomorphic to A_4. Of the elements of H, three are half-turns while eight are threefold rotations. The index of H in G is 10, and the vertices form one H-orbit so that $X = Hx_1$, where x_1 is the vertex at the top indicated in Fig. 4.

We choose $Y = \{(1), (12345), (13524), (14253), (15432), \overline{(14)(23)}, \overline{(13)(45)}, \overline{(12)(35)}, \overline{(25)(34)}, \overline{(15)(24)}\}$, where a bar above the element indicates composition with the inversion $\overline{(1)}$. The last five elements in Y represent mirror reflections. All the right coset representatives in Y fix x_1.

The coloring is obtained using the partition $P = \{hJ_1x_1 \mid h \in H\}$, where $J_1 = \langle(234)\rangle$. The element (234) is the 120° rotation whose axis passes through the center of the face on the right labeled 5 and the center of the opposite face. The elements of the partition P are given by $P = \{J_1x_1, (12)(34)J_1x_1, (13)(24)J_1x_1, (14)(23)J_1x_1\}$, where J_1x_1, $(12)(34)J_1x_1$, $(13)(24)J_1x_1$ and $(14)(23)J_1x_1$ are assigned, respectively, the colors red, green, blue and yellow. The J_1-orbit of x_1, J_1x_1, consists of the three points x_1, $(234)x_1$ and $(243)x_1$, and these points have been colored red in Fig. 1. The elements (12)(34), (13)(24) and (14)(23) represent half-turns or 180° rotations with axes as indicated in Fig. 4. These axes pass through the center of the icosahedron and are mutually perpendicular.

In this example, $S_1 = \bigcup_{y \in Y} J_1 y$. To determine if the coloring is perfect, we check if S_1 is a subgroup of G. However, S_1 has

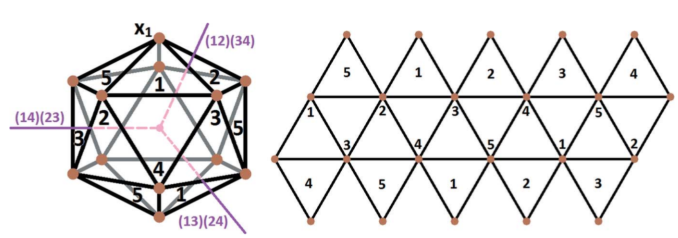


Figure 4
Numbered faces of the icosahedron.

30 elements. Therefore, S_1 cannot be a subgroup of G since G has no subgroup of order 30. We obtain H^* by checking if the right coset representatives in Y induce a permutation of colors. We find that only the identity (1) does, so that the color group $H^* = H$, a subgroup of index 10 in G.

4.2. Regular dodecagon

A regular dodecagon has symmetry group $G = \langle a, b\rangle$ which is isomorphic to D_{12}, where a is the 30° counterclockwise rotation centered at the middle of the dodecagon and b is the reflection along the vertical line passing through the center, as shown in Fig. 5. The set X is the set of slices of the dodecagon, which forms only one G-orbit.

A coloring of the dodecagon is shown in Fig. 6.

$H = \{e, a^6, b, a^6b\}$, $[G:H] = 6$, $X = Hx_1 \cup Hx_2 \cup Hx_3 \cup Hx_4$.

$Y = \{e, a^3, a^2, a, a^2b, ab\}$, where $a^3x_1 = x_2$, $a^2x_1 = x_3$, $ax_1 = x_4$, $a^2bx_1 = x_3$, $abx_1 = x_4$.

$P = \{hJ_1\{x_1, x_4\} \mid h \in H\} \cup \{hJ_2\{x_2, x_3\} \mid h \in H\}$, where $J_1 = \{e, b\}$ and $J_2 = \{e, a^6b\}$.

In this example, $S_1 = J_1 \cup J_1a \cup J_1ab = \{e, a, a^{11}, b, ab, a^{11}b\}$, which is not a subgroup of G. Theorem 2 states that for H^* to be equal to G, it is necessary that $S_1 \leq G$. Therefore, $H^* \neq G$. To get H^*, we check the right coset representatives in Y. Of these, only e and a^3 induce a permutation of colors; therefore $H^* = H \cup Ha^3$, which is an index 3 subgroup of G.

4.3. Cube

A cube has symmetry group G of order 48. Some of its elements are:

(i) a is the 120° clockwise rotation (when viewed from the up–left–front vertex) with axis passing through the up–left–front vertex and the down–right–back vertex.

(ii) b is the 90° clockwise rotation (when viewed from above) with axis passing through the center of the faces above and below the cube.

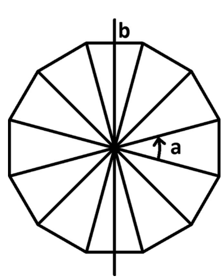


Figure 5
The symmetry group $G = \langle a, b\rangle \cong D_{12}$ of the regular dodecagon.

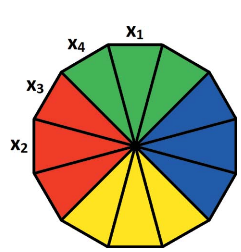


Figure 6
Colored regular dodecagon slices with color group H^* of index 3 in G.

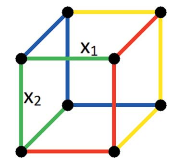


Figure 7
Colored cube edges with color group of index 6 in G.

(iii) c is the half-turn with axis passing through the midpoints of the front-left edge and the back-right edge.

(iv) $\bar{1}$ is the inversion about the center of the cube.

G is isomorphic to $S_4 \times C_2$. We will take the set X to be the set of edges of the cube, which form one orbit under the symmetry group.

For our coloring (Fig. 7), the color group H^* is of index 6 in G.

$H = \langle b, c\rangle \cong D_4$. We have $[G:H] = 6$, $X = Hx_1 \cup Hx_2$.

$Y = \{e, a, b^{-1}a^{-1}, \bar{1}bc, \bar{1}b^2a, \bar{1}cb^2a^{-1}\}$ where $ax_1 = x_2$, $b^{-1}a^{-1}x_1 = x_1$, $\bar{1}bcx_1 = x_1$, $\bar{1}b^2ax_1 = x_2$ and $\bar{1}cb^2a^{-1}x_1 = x_1$.

$P = \{hJ_1\{x_1, x_2\} \mid h \in H\}$, where $J_1 = \{e, c\}$.

For this example, we have $S_1 = \bigcup_{y\in Y} J_1y$. S_1 has 12 elements, two of which are a and $b^{-1}a^{-1}$. However, the product $b^{-1}a^{-1} \cdot a = b^{-1}$ is not an element of S_1. Hence, S_1 is not a subgroup of G. But theorem 2 tells us that S_1 should be a subgroup of G for H^* to be equal to G. Thus, $H^* \neq G$. To find H^*, we take a look at the right coset representatives in Y. Among these, only the identity e induces a permutation of colors, so that the color group is H.

4.4. Planar tiling

Here we consider a tiling of the plane with symmetry group $G = \langle x, y, a\rangle$, where x and y are translations, while a is a half-turn with indicated center, as shown in Fig. 8. The group G is a plane crystallographic group of type $p2$ or 2222 in orbifold notation. The set X is the set of tiles (angels) in the tiling.

For the first coloring (Fig. 9), the color group H^* is G.

$H = \langle x^5, x^2y, a\rangle$. H is also of type $p2$, $[G:H] = 5$, $X = Hx_1 \cup Hx_2 \cup Hx_3 \cup Hx_4 \cup Hx_5$.

$Y = \{e, x, x^{-1}, x^3a, x^2a\}$, where $xx_1 = x_2$, $x^{-1}x_1 = x_3$, $x^3ax_1 = x_4$ and $x^2ax_1 = x_5$.

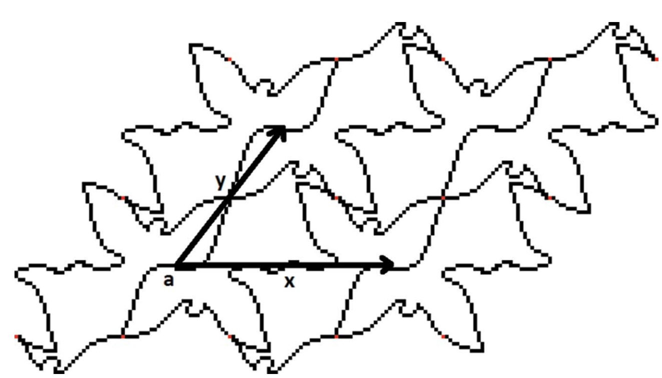


Figure 8
The symmetry group $G = \langle x, y, a\rangle$ of type $p2$ or 2222.

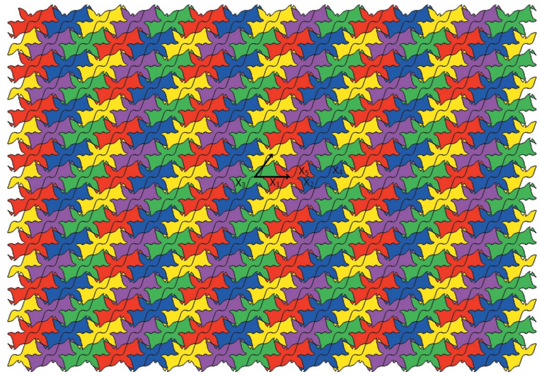


Figure 9
Colored tiling with color group G.

$P = \{hJ_1\{x_1, x_5\} \mid h \in H\} \cup \{hJ_2\{x_2, x_4\} \mid h \in H\} \cup \{hJ_3x_3 \mid h \in H\}$, where $J_1 = \langle x^5, x^2y\rangle$, $J_2 = \langle x^5, x^2y\rangle$ and $J_3 = \langle x^5, x^2y, a\rangle = H$. Here, $\{hJ_1\{x_1, x_5\} \mid h \in H\}$ represents the red and violet angels, $\{hJ_2\{x_2, x_4\} \mid h \in H\}$ represents the blue and yellow angels, while $\{hJ_3x_3 \mid h \in H\}$ is for the green angels.

In this example, $S_1 = J_1 \cup J_1x^2a$, $S_2 = J_2xx^{-1} \cup J_2x^3ax^{-1} = J_2 \cup J_2x^4a$ and $S_3 = J_3$. We have $S_1 \leq G$, $xS_1x^{-1} = S_2$, $x^{-1}S_1x = S_3$ and $(x^3a)S_1(x^3a)^{-1} = S_2$. Thus, from theorem 2, we have $H^* = G$.

For the second coloring (Fig. 10), the color group H^* is H, and is of index 5 in G.

$H = \langle x^5, x^2y, a\rangle$. H is also of type $p2$, $[G : H] = 5$, $X = Hx_1 \cup Hx_2 \cup Hx_3 \cup Hx_4 \cup Hx_5$.

$Y = \{e, x, x^{-1}, x^3a, x^2a\}$, where $xx_1 = x_2$, $x^{-1}x_1 = x_3$, $x^3ax_1 = x_4$ and $x^2ax_1 = x_5$.

$P = \{hJ_1x_1 \mid h \in H\} \cup \{hJ_2x_2 \mid h \in H\} \cup \{hJ_3x_3 \mid h \in H\} \cup \{hJ_4x_4 \mid h \in H\} \cup \{hJ_5x_5 \mid h \in H\}$, where $J_1 = H$, $J_2 = H$, $J_3 = \langle x^5, x^2y\rangle$, $J_4 = \langle x^5, x^2y\rangle$ and $J_5 = \langle x^5, x^4y^2, a\rangle$. There are eight colors in this coloring. The set $\{hJ_1x_1 \mid h \in H\}$ gives us the yellow angels, $\{hJ_2x_2 \mid h \in H\}$ gives us the pink angels, while $\{hJ_3x_3 \mid h \in H\}$ gives us two colors, green and red. The set $\{hJ_4x_4 \mid h \in H\}$ also gives us two colors, blue and orange, and finally, $\{hJ_5x_5 \mid h \in H\}$ gives us the colors gray and violet.

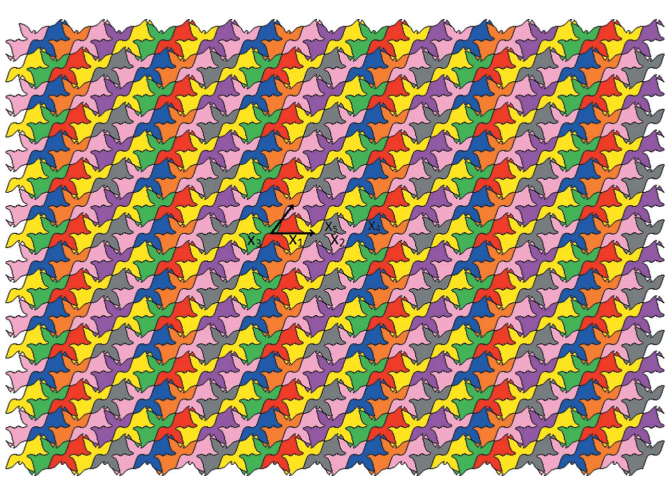


Figure 10
Colored tiling with color group of index 5 in G.

In this example, $S_i = J_i$ for $i = 1, 2, 3, 4, 5$. However, $x^{-1}J_1x$ cannot be equal to J_3, since J_1 contains a half-turn, while J_3 does not, and thus they cannot be conjugate subgroups. But $x^{-1} \in Y_3$ since $x^{-1}x_1 = x_3$. Therefore, we have found an element $y \in Y_3$ which does not satisfy $yS_1y^{-1} = S_3$. Consequently, from theorem 2, we have $H^* \neq G$, and so $H^* = H$.

4.5. Hyperbolic plane tiling

Next, we consider a tiling of the hyperbolic plane. The symmetry group of the tiling is $G = \langle P, Q, R\rangle$ where P, Q and R are reflections with mirror axes shown in Fig. 11. The group G is a triangle group of type *542 generated by reflections on the sides of a triangle with interior angles $\pi/5$, $\pi/4$ and $\pi/2$. The set X is the set of hyperbolic triangles in the tiling.

For our coloring, we consider the subgroup $H = \langle P, Q, RPR, RQRPRQR\rangle$ of index 5 in G. The axes of the reflections RPR and $RQRPRQR$ are also indicated in Fig. 11, along with the representatives for the five H-orbits of X.

For this example, the set Y of right coset representatives is given by $Y = \{e, R, RQ, RQR, RQRQ\}$, where $Rx_1 = x_2$, $RQx_1 = x_3$, $RQRx_1 = x_4$ and $RQRQx_1 = x_5$.

We now present a coloring whose color group is H. We use the partition $P = \{hJ_1x_1 \mid h \in H\} \cup \{hJ_2x_2 \mid h \in H\} \cup \{hJ_3x_3 \mid h \in H\} \cup \{hJ_4x_4 \mid h \in H\} \cup \{hJ_5x_5 \mid h \in H\}$, where each of the J_i's is equal to H. For $i = 1, 2, 3, 4, 5$, we have $S_i = J_i = H$.

To determine the color group, we take a look at the right coset representative R. Note that R is a reflection, so that $R^{-1} = R$. Since $Rx_1 = x_2$, we have $R \in Y_2$. Thus, we should have $RS_1R = S_2$ for the coloring to be perfect, according to theorem 2. However, consider $Q \in S_1$. The element RQR cannot be an element of S_2, since $S_2 = H$ and RQR is in the

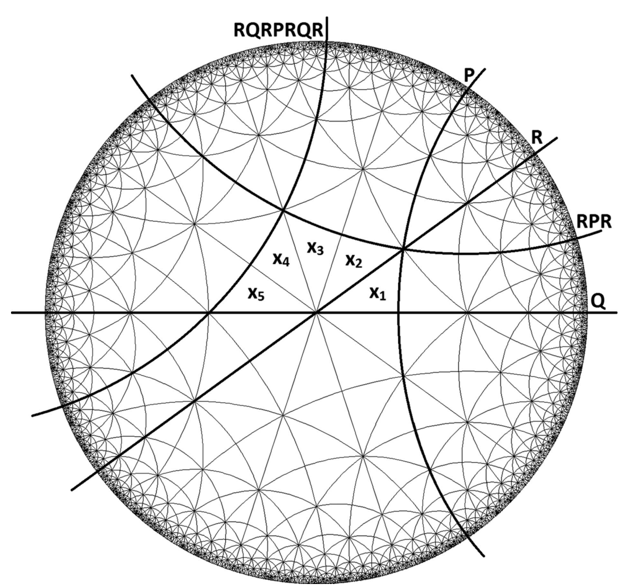


Figure 11
The symmetry group $G = \langle P, Q, R\rangle$ and its subgroup $H = \langle P, Q, RPR, RQRPRQR\rangle$.

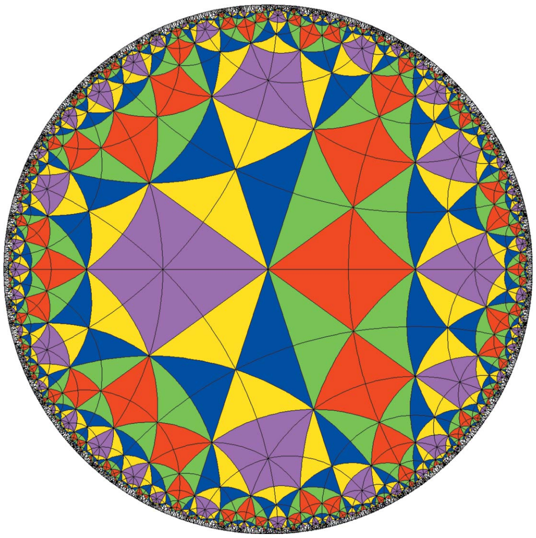


Figure 12
Colored hyperbolic tiling with color group of index 5 in G.

coset $HRQR$, which is not equal to H. Hence, $H^* \neq G$, and so $H^* = H$. The coloring appears in Fig. 12.

4.6. Six-pointed star

The six-pointed star in Fig. 13 has symmetry group $G = \langle a, b\rangle$ isomorphic to D_6, the dihedral group with 12 elements. Here, a is the 60° counterclockwise rotation centered at the middle of the star, while b is the reflection whose axis is the horizontal line passing through the middle of the star. The set $\mathcal{X}$ of objects to be colored is the set of vertices of the star. This set forms two G-orbits. The first G-orbit is the set of outer vertices, while the second G-orbit is the set of inner vertices. We write $\mathcal{X} = Gv \cup Gx$, where v and x are the vertices indicated in Fig. 13.

To color the vertices according to our framework, we need to color each G-orbit separately. For each of the following colorings, we have $H = G$ and $J = \langle a^3, b\rangle$. Note that $H = G$ implies $Y = \{e\}$.

The first coloring is a perfect coloring. The outer vertices are colored using the partition $P_o = \{hJv \mid h \in H\} = \{Jv, aJv, a^2Jv\}$, where Jv is colored green, aJv is red and a^2Jv is blue. Meanwhile, the inner vertices are colored using $P_i = \{hJx \mid h \in H\} = \{Jx, aJx, a^2Jx\}$, where Jx is green, aJx is red

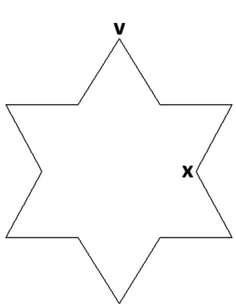


Figure 13
A six-pointed star with symmetry group G isomorphic to D_6.

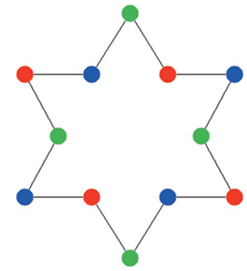


Figure 14
A perfect coloring of the vertices of the six-pointed star.

and a^2Jx is blue. For both P_o and P_i, it is clear that $H^* = G$ since $H = G$ and H^* is a subgroup of G which contains H. In other words, both P_o and P_i induce perfect colorings of their respective G-orbits. When combined into a single figure (Fig. 14), the vertices of the star become perfectly colored.

We now proceed to the second coloring. For this coloring, we make use of the same P_o and P_i, except that, in this example, we color aJx with blue and a^2Jx with red. Since we used the same partitions, both the outer and inner vertices are still perfectly colored. However, when we combine them into a single figure (Fig. 15), we do not get a perfect coloring. Instead, the color group of the coloring is $\{e, a^3, b, a^3b\}$, which is of index 3 in G.

For the third coloring, we again make use of the same partitions P_o and P_i as in the first coloring. We also color the inner vertices in exactly the same manner as in the first example, but we color the outer vertices differently. Here, we color Jv red, aJv is blue while a^2Jv is colored green. Again, both the outer and inner vertices are perfectly colored, but when combined into one figure (Fig. 16), we fail to get a perfect coloring. Instead, we see that the color group is the subgroup generated by a, which is of index 2 in G.

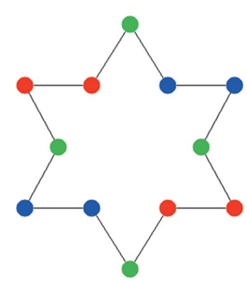


Figure 15
A coloring of the vertices of the six-pointed star with color group of index 3 in G.

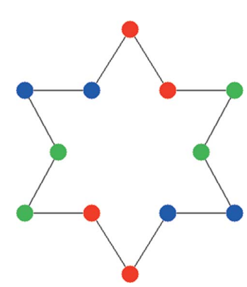


Figure 16
A coloring of the vertices of the six-pointed star with color group of index 2 in G.

These examples stress the importance of the requirement that the symmetry group G should act transitively on the set of objects to be colored for us to be able to use theorem 2. For colorings of sets that form more than one G-orbit, the best that the theorem can do is obtain the color group of each G-orbit, and not the coloring as a whole. Furthermore, the second and third examples illustrate that it is not always enough to simply take the intersection of the color groups we obtained for each G-orbit to get the color group of the whole figure.

In general, if there is no color sharing between G-orbits, it is enough to find the intersection of the color groups of each G-orbit to get the color group of the entire coloring. However, when there is color sharing between G-orbits, there are instances when it is not enough to just take the intersection of the color groups of the G-orbits.

5. Conclusion and recommendations

In this paper, we constructed colorings of symmetrical patterns where the color group always contains a chosen subgroup H of the symmetry group G. The main result gives us a framework for determining the color groups of the colorings. Using this framework, we have obtained the color groups of different colorings and we have seen how the indices of these color groups in the symmetry group G vary from coloring to coloring.

We were able to formulate theorem 2, which completely solves the problem of finding the color group when the index of H in G is prime. Unfortunately, when the index of H in G is not prime, the computations in the framework become tedious, especially when the index becomes large. This is due to the possible existence of intermediate subgroups between H and G. Further research is necessary to simplify the framework for the case when the index of H in G is not prime.

Through the latter examples, we have seen the importance of the restriction that G should act transitively on the set of objects being colored. We observed that, in general, it is not enough to simply take the intersection of the color groups obtained for each of the G-orbits. Methods for obtaining color groups of colorings where the symmetry group does not act transitively on the set of objects to be colored could be a good topic for research.

Another possible research topic is the color-fixing group of the coloring. It is beyond the scope of this paper, but the main theorem, or more specifically the proof, might give some clues on how to identify this group. Recall that the S_i's are stabilizers of some specific color (*e.g.* S_1 is the stabilizer of the color J_1X_1), so these stabilizers might prove useful in the study of the color-fixing group.

Finally, the question of equivalence of colorings obtained using this framework should be of interest, as two colorings with the same color group need not be equivalent.

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